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Theoretical analysis of the steady state particle size distribution in limited breakage processes

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Abstract. The steady state particle size distribution is examined, resulting from a breakage process with a maximum stable size and a homogeneous continuous kernel. The dynamic breakage problem is transformed into one that allows direct solutions for the steady state distribution. The latter depends on the breakage kernel and on the ratio of critical to initial size. As this ratio goes to zero the steady state distribution approaches its limiting form obtained by the authors previously [7]. A general theoretical analysis concerning the steady state distribution is presented herein. The asymptotic behaviour is determined with regard to various limits. Perturbation analysis for nearly uniform kernels reveals several interesting features of the problem. For the general continuous kernel, the problem can be cast in a matrix form amenable to a conventional theoretical treatment. Finally, comparisons of the new results with existing solutions of the dynamic problem, for large times, confirm their validity.

1. Introduction

Breakage, alternatively referred to as fragmentation, is of great interest to many branches of science and engineering. For instance some physical aspects of fragmentation are recognized in the atomic collision cascades, in energy cascades of turbulence and in the multivalley structure of the phase space of disordered systems. Technological aspects of fragmentation concern mineral processing, polymer degradation and break-up of liquid droplets or air bubbles. Even every day problems such as car parking [1] can be cast in the form of a fragmentation problem. A rather extensive account for processes based on fragmentation is presented by Redner [2]. The term breakage is considered synonymous to fragmentation and used mainly in fluid processes while fragmentation refers to solids.

In many situations involving fragmentation, a critical particle size X_m may exist below which there is no further break-up. This is widely accepted for certain physical systems such as the turbulent flow of liquid–liquid dispersions [3]; indeed it is considered that the turbulent flow field cannot cause breakage of droplets below a certain size related to the turbulent eddy structure. On the other hand, forces acting on large droplets (of size greater than X_m) can lead to daughter particles much smaller than X_m. One may call this type of size reduction *limited breakage*. It is noted that the small cut-off size considered by Redner [2] does not actually lead to a limited breakage mechanism because that restriction is on the ratio of daughter to parent particle and not on the absolute size of the parent particle. An obvious difference between continuous and limited breakage is that in the former the particle size distribution keeps changing with time, while in the latter the process results in a steady state. It is worth noting that the existence of a critical size in limited

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breakage excludes shattering phenomena which may be important in the study of continuous (unlimited) breakage processes.

The existence of a critical size X_m in limited breakage, implies that all the particles in the steady state size distribution are smaller then X_m. If only breakage takes place, this steady state may be called *static*, in the sense that particles smaller than the critical size remain unchanged, and additionally it may depend in principle on the earlier states of the system. By contrast, a *dynamic* steady state resulting from a combination of two or more competing mechanisms, e.g. breakage and coalescence [4], is characterized by a ceaseless alteration of particles. The dynamic steady state is independent of earlier states of the system and thus from the initial conditions.

Although there is an extensive literature on the determination of the critical size for the limited breakage, as a function of the physicochemical characteristics of the system, there are few solutions available, mainly numerical, of the mathematical problem [5, 6]. Such theoretical treatments may be illuminating in general, and helpful in addressing issues of practical interest. Recently Kostoglou and Karabelas [7] showed that the steady state size distribution for continuous homogeneous kernels takes an asymptotic form as the ratio of critical to initial particle size approaches zero. This paper presents a significant extension and generalization of previous work by examining the steady state size distribution with no restriction on the value of this ratio, i.e. for finite values as well.

The structure of this paper is as follows. First (section 2) the general formulation for the limited breakage problem is outlined. Reduction of the dynamic problem to the steady state is analysed in section 3. Section 4 includes several asymptotic results for the steady state size distribution and related auxiliary functions. A general perturbation analysis for a slightly perturbed uniform kernel follows (section 5); many useful results are deduced from this analysis. A more general class of kernels (which at least in principle represents all continuous kernels) is treated next (section 6); there the results of the previous sections are employed to study the approach of the steady state size distribution to its asymptotic form (limiting steady state) as the ratio of critical to initial particle size decreases. Finally, the direct solutions for the steady state are compared with solutions that are obtained by using existing analyses of the dynamic unlimited breakage.

2. Mathematical formulation

The evolution of size distribution of dispersed particles for limited breakage is given by

$$\frac{\mathrm{d}f'(x',t)}{\mathrm{d}t'} = \int_{x'}^{\infty} v'(y')p'(x',y)b'(y)f'(y,t)\,\mathrm{d}y - b'(x')f'(x',t) \qquad x' > X_m \tag{1a}$$

$$\frac{\mathrm{d}f'(x',t)}{\mathrm{d}t'} = \int_{X_m}^{\infty} v'(y')p'(x',y)b'(y)f'(y,t)\,\mathrm{d}y \qquad x' < X_m \tag{1b}$$

where X_m is the critical size below which particles do not suffer breakage, t' time, x' particle volume, $f'(x',t)$ the particle number density distribution, $b'(x')$ the breakage probability of particles of volume x', $v'(y')p'(x',y')$ is the distribution of particles of volume x' resulting from the break-up of a particle of volume y', $v'(y')$ the number of particles resulting from the break-up of a particle of volume y', and X_m is the critical size below which particles do not suffer breakage.

Let $f'_0(x') = f'(x',0)$ be the initial distribution. The total volume concentration, the total number concentration and the mean size of the initial distribution are, respectively:

$$M = \int_0^{\infty} x f'_0(x)\,\mathrm{d}x$$

$$N_0 = \int_0^\infty f_0'(x)\,\mathrm{d}x$$

$$x_0 = \frac{M}{N_0}.$$

The functions and variables already introduced can be expressed in dimensionless form, as follows:

$$x = \frac{x'}{x_0} \qquad y = \frac{y'}{x_0} \qquad \tau = b'(x_0)\cdot t \qquad x_m = \frac{X_m}{x_0} \qquad b(x) = \frac{b'(x)}{b'(x_0)}$$

$$f(x,\tau) = \frac{x_0 f'(x',t)}{N_0} \qquad p(x,y) = x_0 p'(x',y') \qquad v(y) = v'(y')$$

and equations (1*a*, *b*) can be written as

$$\frac{\mathrm{d}f(x,\tau)}{\mathrm{d}\tau} = \int_x^\infty v(y)p(x,y)b(y)f(y,\tau)\,\mathrm{d}y - b(x)f(x,\tau) \qquad x > x_m \tag{2a}$$

$$\frac{\mathrm{d}f(x,\tau)}{\mathrm{d}\tau} = \int_{x_m}^\infty v(y)p(x,y)b(y)f(y,\tau)\,\mathrm{d}y \qquad x < x_m. \tag{2b}$$

There is a large number of solutions to the above problem available for the case $x_m = 0$. For certain simple forms of functions $b(x)$ and $v(y)p(x,y)$ analytical solutions exist [8–10]. Similarity transformations [11–13] can be used if the above functions satisfy certain requirements. Finally, for a general form of kernels, there are specialized numerical methods [14] and Monte Carlo simulations [15]. Because of the linearity of the problem the solution for an arbitrary initial distribution can be obtained from the superposition of solutions for monodisperse initial distributions.

3. Equation for the steady state

The steady state distribution can be obtained from the solution of equations (2) in the limit $\tau = \infty$. Since, this method is quite impractical computationally, one may proceed in a different way.

The following function is introduced:

$$L(x) = b(x)\int_0^\infty f(x,\tau)\,\mathrm{d}\tau \tag{3}$$

which represents the total number of particles with volume x, that suffer breakage during the entire process. This transformation essentially eliminates the breakage frequency.

Integrating equations (2*a*, *b*) from $\tau = 0$ to ∞, one obtains

$$-f_0(x) = \int_x^\infty v(y)p(x,y)L(y)\,\mathrm{d}y - L(x) \qquad x > x_m \tag{4a}$$

$$f_s(x) = \int_{x_m}^\infty v(y)p(x,y)L(y)\,\mathrm{d}y + f_0(x) \qquad x < x_m \tag{4b}$$

where $f_s(x)$ is the dimensionless steady state particle size distribution. The function $L(x)$ depends on $v(y)p(x,y)$ and $f_0(x)$, whereas the steady state size distribution $f_s(x)$ depends additionally on x_m. The independence of the steady state size distribution from breakage rate suggests that a steady state exists even for rates which lead the unlimited breakage problem to a shattering-type behaviour.

One may assume that the breakage kernel is independent of the absolute parent particle size but that depends only on the ratio x/y. Volume conservation considerations imply

that the kernel has the form $v(y)p(x, y) = \varphi(x/y)/y$. This type of kernel has been used extensively in the literature; diverse applications include prototype models for the solution of the breakage equation [8–13] and empirical expressions for fitting experimental data [16]. To proceed one should modify the nondimensionalization of particle volume and of steady size distribution to render it independent of the initial distribution:

$$\bar{x} = \frac{x}{x_m} = \frac{x'}{X_m} \qquad \bar{f}_s(\bar{x}) = \frac{X_m^2 f_s'(x')}{M} = x_m^2 f_s(x). \tag{5}$$

In the linear equations (4), $f_0(x)$ can be interpreted as a kind of driving force. Indeed, considering an elementary form of the initial distribution such as $f_0(x) = \delta(x-1)$ a Green function $\bar{f}_{sg}(\bar{x}; x_m)$ results. For an arbitrary initial distribution $f_0(x)$, application of the superposition principle leads to

$$\bar{f}_s(\bar{x}) = \int_{x_m}^{\infty} y f_0(y) \bar{f}_{sg}\left(\bar{x}; \frac{x_m}{y}\right) \mathrm{d}y + x_m^2 f_0(\bar{x} x_m). \tag{6}$$

The second term in the right-hand side of the above equation represents the part of the initial distribution with sizes smaller than X_m, which remains unaltered during the breakage process. In view of equation (6), it is not restrictive to assume that the initial distribution is monodisperse $\delta(x-1)$; any result obtained with the latter can be directly generalized for arbitrary initial distribution via equation (6). Using the monodisperse initial condition and the new function $q(x) = L(x) - \delta(x-1)$, equations (4) are modified as

$$\varphi(x) + \int_x^1 \frac{1}{y}\varphi(x/y) q(y)\, \mathrm{d}y - q(x) = 0 \tag{7a}$$

$$\bar{f}_s(\bar{x}) = x_m^2 \int_{x_m}^1 \frac{1}{y}\varphi\left(\frac{\bar{x}}{y} x_m\right) q(y)\, \mathrm{d}y + x_m^2 \varphi(\bar{x} x_m). \tag{7b}$$

Equation (7) is a form of Volterra equation of the second kind, that can be solved using the well known method given by Tricomi [17], i.e.

$$\begin{aligned} q(x) &= \varphi(x) + \int_x^1 H(x, y)\varphi(y)\, \mathrm{d}y \\ H(x, y) &= \sum_{n=0}^{\infty} A_{n+1}(x, y) \\ A_{n+1}(x, y) &= \int_x^y A(x, z) A_n(z, y)\, \mathrm{d}z \qquad n = 1, 2, 3, \ldots, \infty \\ A_1(x, y) &= A(x, y) = \frac{1}{y}\varphi(x/y). \end{aligned} \tag{8}$$

This solution is quite impractical for computations; thus it is preferable to proceed in different ways in order to solve the equations (7) and to study the solutions.

4. General behaviour—asymptotic results

It is observed that the behaviour of the distribution $\bar{f}_s(\bar{x})$ with respect to the variable x_m resembles the temporal evolution of a distribution. To demonstrate and exploit this analogy, the new timelike variable $t = -\log(x_m)$ is defined and the alternative symbolism $\bar{f}_s(\bar{x}; t)$ is used to denote the functional dependence on t. The initial condition (for $t = 0$), easily obtained from equation (7*b*), is $\bar{f}_s(\bar{x}; 0) = \varphi(\bar{x})$. As 'time' t increases the distribution 'evolves'. If the function $\varphi(x)$ is continuous in $(0, 1]$ then, as Kostoglou and Karabelas

[7] have shown, a limiting steady state is reached after sufficient 'time' has passed (i.e. $x_m \ll 1$). This final steady state is given as

$$\bar{f}_s(\bar{x};\infty) = \frac{A}{\bar{x}^2}\int_0^{\bar{x}} z\varphi(z)\,\mathrm{d}z \qquad \text{where } A = -\left(\int_0^1 z\ln(z)\varphi(z)\,\mathrm{d}z\right)^{-1}. \tag{9}$$

The evolution of the distribution $\bar{f}_s(\bar{x};t)$ between the initial condition and the limiting steady state concerns the present work.

4.1. Behaviour of $q(x)$ for small x

It has been shown [6] that, for continuous kernels $\varphi(x), q(x) \propto x^{-2}$ in the limit $x \to 0$. This behaviour creates serious difficulties in the direct numerical solution of equation (7*a*). These difficulties can be easily overcome by taking into account the above behaviour to make smooth the unknown function, i.e. by multiplication with x^2.

4.2. Behaviour of $\bar{f}_s(\bar{x})$ for small $\bar{x}$

If, in the limit $x \to 0$, $\varphi(x) \propto x^n$ (where $n > -2$ from mass conservation considerations) then a simple substitution into equation (7*b*) leads to $\bar{f}_s(\bar{x}) \propto \bar{x}^n$ for $\bar{x} \to 0$. This statement can be obviously extended for every finite interval $[0, a]$, where $a < 1$, for which $\varphi(x)$ has a power form.

4.3. Behaviour of $q(x)$ for large $x (x \to 1)$

We define the perturbation variable $\varepsilon = 1 - x$. It is interesting that the series solution (8) converges as ε^n. Although this iterative solution can be used to find $q(x)$ as a power series of ε, there is a more systematic way, as follows. Equation (7*a*) can be transformed into

$$\varphi(1-\varepsilon) + \varepsilon\int_0^1 \frac{1}{1-\varepsilon y}\varphi\left(\frac{1-\varepsilon}{1-\varepsilon y}\right) q(1-\varepsilon y)\,\mathrm{d}y - q(1-\varepsilon) = 0.$$

If $\varphi(x)$ is analytical in $(0, 1]$ then it can be written as a MacLaurin series around 1 with respect to ε. Assuming that $q(1-\varepsilon) = \sum_{i=0}^{\infty} q_i\varepsilon^i$, carrying out the expansions and integrations and equating coefficients of equal powers of ε, in the above equation, the coefficients q_i can be found as functions of the derivatives of $\varphi(x)$ at $x = 1$. The above procedure is valid up to $i = n$ for kernels with continuous derivatives up to nth order. The result for the zero- and first-order term of this expansion is

$$q(1-\varepsilon) = \varphi(1) + (\varphi^2(1) - \varphi'(1))\varepsilon. \tag{10}$$

4.4. Behaviour of $\bar{f}_s(\bar{x};t)$ for small t

If $\varepsilon = 1 - x_m = 1 - \mathrm{e}^{-t}$, equation (7*b*) can be transformed in the form

$$\bar{f}_s(\bar{x}) = (1-\varepsilon)^2\varepsilon\int_0^1 \frac{1}{1-\varepsilon y}\varphi\left[\frac{\bar{x}(1-\varepsilon)}{1-\varepsilon y}\right] q(1-\varepsilon y)\,\mathrm{d}y + (1-\varepsilon)^2\varphi(\bar{x}(1-\varepsilon)).$$

For $\varphi(x)$ analytical in $(0, 1]$ the terms of the above equation can be expanded in power series of ε. Using the series for $q(x)$ referred to above, after integration and collection of terms of the same order, a result of the following type is obtained

$$\bar{f}_s(\bar{x}) = \sum_{i=0}^{\infty} \bar{f}_i(\bar{x})\varepsilon^i$$

where the functions $\bar{f}_i(\bar{x})$ are combinations of the derivatives $\varphi^{(j)}(\bar{x})$ and $\varphi^{(j)}(1)$ for $j < i$. This procedure is valid up to $i = n$ for a kernel $\varphi(x)$ with derivatives continuous up to nth order. The result for the zero-, first- and second-order terms of the expansion is

$$\bar{f}_s(\bar{x}) = \varphi(\bar{x}) + [\varphi(\bar{x})\varphi(1) - 2\varphi(\bar{x}) - \varphi'(\bar{x})]\varepsilon + \left[\varphi(\bar{x})\left(\frac{\varphi^2(1)}{2} - \frac{\varphi'(1)}{2} - \frac{3\varphi(1)}{2} + 1\right) + \bar{x}\varphi'(\bar{x})\left(\frac{\varphi(1)}{2} + 2\right) + \frac{1}{2}\bar{x}^2\varphi''(\bar{x})\right]\varepsilon^2. \tag{11}$$

5. Analysis for nearly uniform binary breakage

A case of theoretical interest is the binary breakage with nearly uniform kernel. It means that some pairs of daughter particles appear with slightly larger probabilities than others. The nearly uniform breakage kernel has the following form

$$\varphi(z) = 2(1 + \varepsilon\rho(z)) \tag{12}$$

where $\rho(z)$ is of order 1 and $\varepsilon \ll 1$. To meet the requirements of binary breakage and mass conservation the function $\rho(z)$ must have the following properties:

$$\text{(i) } \rho(z) = \rho(1 - z) \qquad \text{(ii) } \int_0^{0.5} \rho(z)\,\mathrm{d}z = 0. \tag{13}$$

For $\varepsilon < 1$ the series $q(x) = \sum_{i=0}^{\infty} \varepsilon^i q_{(i)}(x)$ converges. The functions $q_{(i)}(x)$ are unknown and must be computed from equation (7*a*). Substituting the above series in equation (7*a*) and collecting terms of equal powers of ε, the following hierarchy of equations is obtained:

$$2 + \int_x^1 \frac{2}{y} q_{(0)}(y)\,\mathrm{d}y - q_{(0)}(x) = 0 \qquad \text{for } i = 0 \tag{14a}$$

$$2\rho(x)\delta(i-1) + \int_x^1 \left[\frac{2}{y} q_{(i)}(y) + \frac{2}{y}\rho\left(\frac{x}{y}\right) q_{(i-1)}(y)\right]\mathrm{d}y - q_{(i)}(x) = 0 \tag{14b}$$

for $i = 1$ to ∞.

Equation (14*a*) corresponds to the case of uniform breakage (reference state) and has the simple solution $q_{(0)}(x) = 2x^{-2}$. This relation is substituted in the first ($i = 1$) of equations (14*b*); then, equations (14*b*) are differentiated with respect to x and the following linear differential equations result ($i = 1, 2, \ldots, \infty$):

$$q'_{(i)}(x) + \frac{2}{x} q_{(i)}(x) - 2\rho'(x)\delta(i-1) + \frac{2}{x}\rho(1) q_{(i-1)}(x) - \int_x^1 \frac{2}{y^2}\rho'\left(\frac{x}{y}\right) q_{(i-1)}(y)\,\mathrm{d}y = 0. \tag{15}$$

The initial conditions for these equations are easily determined from equations (14*b*) in the limit $x = 1$; i.e. $q_{(i)}(1) = 2\rho(1)\delta(i-1)$. Integration of equations (15) with the above initial conditions leads to

$$q_{(1)}(x) = \frac{2}{x^2}\left[\rho(1) - 2\rho(1)\log(x) - \int_x^1 y^2\rho'(y)\,\mathrm{d}y - 2\int_x^1\int_z^1 \frac{z^2}{y^4}\rho'\left(\frac{z}{y}\right)\mathrm{d}y\,\mathrm{d}z\right] \qquad i = 1 \tag{16a}$$

$$q_{(i)}(x) = \frac{2}{x^2}\left[\int_x^1 y\rho(1) q_{(i-1)}(y)\,\mathrm{d}y - \int_x^1\int_z^1 \frac{z^2}{y^2}\rho'\left(\frac{z}{y}\right) q_{(i-1)}(y)\,\mathrm{d}y\,\mathrm{d}z\right] \qquad i = 2 \text{ to } \infty. \tag{16b}$$

$q_{(1)}(x)$ is given directly as a function of kernel by equation (16*a*).

The other functions $q_{(i)}(x)(i \geqslant 2)$ must be successively computed from equations (16*b*). Another interesting quantity is the coefficient C that is the limit of $x^2 q(x)$ as $x \to 0$. For the kernel (12) it can be expanded as a perturbation series $C = 2(1 + \sum_{i=1}^{\infty} C_i \varepsilon^i)$. The coefficient C_1 can be determined from equation (16*a*) by taking the limit $x = 0$. After considerable algebra and using the identity $\int_0^1 y^2 \rho'(y)\,\mathrm{d}y = \rho(1)$, which stems from the mass conservation property of the kernel, the following remarkably simple relation for the coefficient C_1 results

$$C_1 = \int_0^1 y \log(y) \rho(y)\,\mathrm{d}y. \tag{17}$$

A perturbation expansion for the steady state size distribution $\bar{f}_s(\bar{x}) = \sum_{i=0}^{\infty} \varepsilon^i \bar{f}_{s(i)}(\bar{x})$ also converges. After substitution of the kernel (12) and the series expansions of $q(x)$ and $\bar{f}_s(\bar{x})$ into equation (7*b*) the following result for the functions $\bar{f}_{s(i)}(\bar{x})$ is obtained:

$$\bar{f}_{s(0)}(\bar{x}) = 2 \qquad i = 0 \text{ (uniform breakage)} \tag{18a}$$

$$\bar{f}_{s(1)}(\bar{x}) = 2x_m^2 \left[\int_{x_m}^1 \left(\frac{1}{y} q_{(1)}(y) + \frac{2}{y^3} \rho\left(\frac{\bar{x}}{y} x_m \right) \right) \mathrm{d}y + \rho(\bar{x} x_m) \right] \qquad i = 1 \tag{18b}$$

$$\bar{f}_{s(i)}(\bar{x}) = 2x_m^2 \int_{x_m}^1 \left(\frac{1}{y} q_{(i)}(y) + \frac{1}{y} \rho\left(\frac{\bar{x}}{y} x_m \right) q_{(i-1)}(y) \right) \mathrm{d}y \qquad i = 2 \text{ to } \infty. \tag{18c}$$

As outlined above, the functions $q_{(i)}(x)$ are determined from equations (16).

There is another more convenient way to compute the first-order perturbation function for the steady state $\bar{f}_{s(1)}(\bar{x})$ with no need for an auxiliary function $q_{(1)}(x)$. Mass conservation requires that

$$\int_0^1 \bar{x} \bar{f}_{s(i)}(\bar{x})\,\mathrm{d}\bar{x} = 0 \qquad \text{for } i = 1 \text{ to } \infty. \tag{19}$$

Equation (18*b*) is substituted in equation (19) to obtain

$$\int_{x_m}^1 \frac{1}{y} q_{(1)}(y)\,\mathrm{d}y = -4 \int_0^1 \int_{x_m}^1 \frac{\bar{x}}{y^3} \rho\left(\frac{\bar{x}}{y} x_m \right) \mathrm{d}y - 2 \int_0^1 \bar{x} \rho(\bar{x} x_m)\,\mathrm{d}\bar{x}. \tag{20}$$

Combining equations (18*b*) and (20) the final result for the first-order perturbation steady state size distribution is given as

$$\bar{f}_{s(1)}(\bar{x}) = 2x_m^2 \left[\int_{x_m}^1 \frac{2}{y^3} \rho\left(\frac{\bar{x}}{y} x_m \right) \mathrm{d}y + \rho(\bar{x} x_m) - 4 \int_0^1 \int_{x_m}^1 \frac{\bar{x}}{y^3} \rho\left(\frac{\bar{x}}{y} x_m \right) \mathrm{d}y\,\mathrm{d}\bar{x} \right.$$
$$\left. -2 \int_0^1 \bar{x} \rho(\bar{x} x_m)\,\mathrm{d}\bar{x} \right]. \tag{21}$$

The significance of this equation is that it relates the steady state directly with the kernel, with no need for an elaborate auxiliary function $q_{(1)}(x)$ as in equation (18*b*). The above procedure can be extended for arbitrary i but the only advantage in this case is the reduction of order of dependence of $\bar{f}_{s(i)}(\bar{x})$, i.e. from $q_{(i)}(x)$ to $q_{(i-1)}(x)$:

$$\bar{f}_{s(i)}(\bar{x}) = 2x_m^2 \left[\int_{x_m}^1 \frac{1}{y} \rho\left(\frac{\bar{x}}{y} x_m \right) q_{(i-1)}(y)\,\mathrm{d}y - 2 \int_0^1 \int_{x_m}^1 \frac{\bar{x}}{y} \rho\left(\frac{\bar{x}}{y} x_m \right) q_{(i-1)}(y)\,\mathrm{d}y\,\mathrm{d}\bar{x} \right]. \tag{22}$$

Table 1. First-order perturbation C_1 of the $\lim_{x=0}[x^2 q(x)]$ for a $n/2$ period cosine perturbation of the uniform kernel.

n	C_1
2	0.247
4	0.078 9
6	0.039 6
8	0.024 1
10	0.016 3
12	0.018 4
20	0.004 78
30	0.002 30
40	0.001 37
50	0.000 91

5.1. Results for harmonic perturbation

In order to obtain numerical results, a specific kernel $\rho(z)$ has to be assumed. For example, the harmonic function $\rho(z) = \cos(n\pi z)$ is used here with $n = 2, 4, 6, \ldots$. The major advantage of this particular form is that every kernel can be expanded in a Fourier series as

$$\rho(z) = \sum_{i=1}^{\infty} a_i \cos(in\pi z). \tag{23}$$

If the first-order perturbation functions for the simple harmonic kernel are $q_{(1)}(x, n)$ and $\bar{f}_{s(1)}(\bar{x}, n)$, the corresponding functions for the general kernel (23) are, respectively,

$$q_{(1)}(x) = \sum_{i=1}^{\infty} a_i q_{(1)}(x, in) \qquad \bar{f}_{s(1)}(\bar{x}) = \sum_{i=1}^{\infty} a_i \bar{f}_{s(1)}(\bar{x}, in). \tag{24}$$

For the case $\rho(z) = \cos(n\pi z)$ the coefficient C_1 (equation (17)) is given as

$$C_1 = \int_0^1 z \log(z) \cos(n\pi z)\, \mathrm{d}z. \tag{25}$$

Values of C_1 which are obtained by numerical integration of the integral in equation (25) for several n values are shown in table 1. Obviously C_1 is a decreasing function of n which means that as n increases the asymptotic behaviour of $q(x)$ for $x \to 0$ gets closer to that of the unperturbed kernel.

Substitution of the particular kernel into equation (18*b*) leads to the following result for the steady state size distribution

$$\begin{aligned}\bar{f}_{s(1)}(\bar{x}, n) = {} & \frac{4\cos(n\pi\bar{x})}{(n\pi\bar{x})^2} - \frac{4\cos(n\pi\bar{x}x_m)}{(n\pi\bar{x})^2} + \frac{4\sin(n\pi\bar{x})}{n\pi\bar{x}} - \frac{4x_m \sin(n\pi\bar{x}x_m)}{n\pi\bar{x}} + \frac{4}{(n\pi)^2} \\ & - \frac{8}{(n\pi)^2}\int_0^1 \frac{\cos(n\pi y) - \cos(n\pi y x_m)}{y}\,\mathrm{d}y - \frac{4\cos(n\pi x_m)}{(n\pi)^2} - \frac{4x_m \sin(n\pi x_m)}{n\pi} \\ & + 2x_m^2 \cos(n\pi\bar{x}x_m).\end{aligned} \tag{26}$$

By taking the limit of this equation at $x_m = 0$ the limiting steady state size distribution results

$$\bar{f}_{s1(1)}(\bar{x}, n) = \frac{4(\cos(n\pi\bar{x}) - 1)}{(n\pi\bar{x})^2} + \frac{4\sin(n\pi\bar{x})}{n\pi\bar{x}} - \frac{8}{(n\pi)^2}\int_0^1 \frac{\cos(n\pi y) - 1}{y}\,\mathrm{d}y. \tag{27}$$

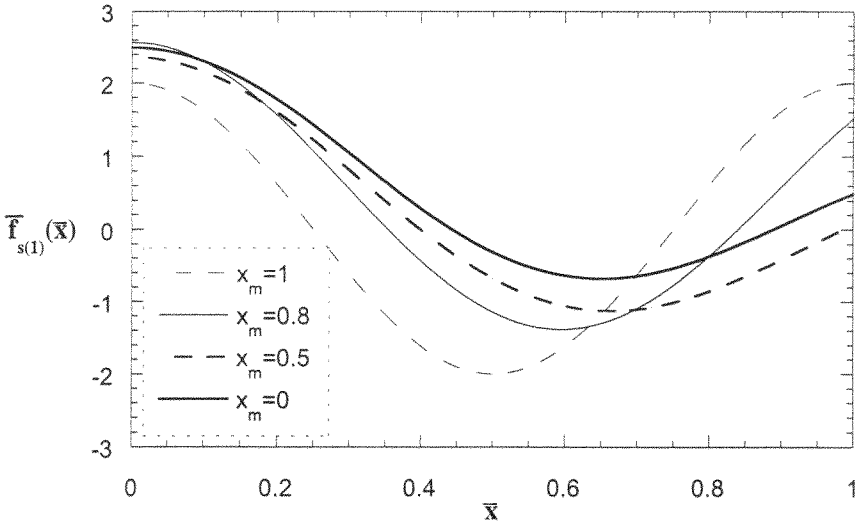

Figure 1. First-order perturbation of the steady state distribution $\bar{f}_{s(1)}(\bar{x})$ for a one period ($n = 2$) cosine perturbation of the uniform kernel, for several x_m values.

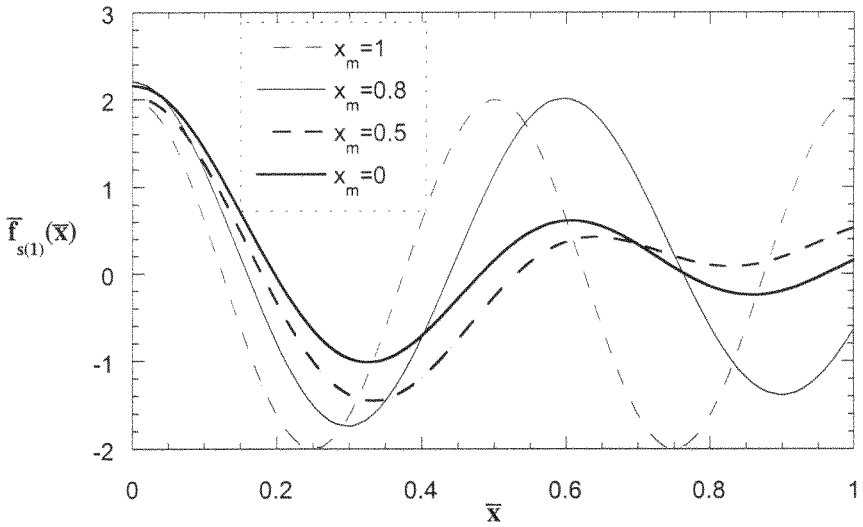

Figure 2. First-order perturbation of the steady state distribution $\bar{f}_{s(1)}(\bar{x})$ for a two period ($n = 4$) cosine perturbation of the uniform kernel, for several x_m values.

A confirmation of the correctness of the entire procedure is that exactly the same result is obtained by substituting the kernel into equation (9) that gives directly the limiting steady state, and expanding the denominator with respect to ε. The perturbation $\bar{f}_{s(1)}(\bar{x})$ in the steady state size distribution for $n = 2$ and $n = 4$ is shown in figures 1 and 2, respectively, for several x_m values. In both cases the limiting steady state is practically reached for $x_m = 0.2$.

6. Analysis for the sum of powers kernel

The *sum of powers* type of kernel comprises a very important class of breakage kernels. The functional form of these kernels is

$$\varphi(z) = \sum_{i=1}^{n} c_i z^{k_i} \tag{28}$$

where $k_i \in (-2, \infty)$ and $c_i \in R^*$. Of course the coefficients c_i must be such as to conserve the total mass, that is $\sum_{i=1}^{n} \frac{c_i}{k_i+2} = 1$. There is a variety of members in this family, from purely empirical kernels that closely fit experimental data [16] to theoretical kernels that admit analytical solutions of the breakage equation (1) [8–10, 13]. Also the entire spectrum of kernels presented by Hill and Ng [18], based on the statistics of multiple breakage, can be written in the above form. In fact, (at least in principle) any kernel continuous in $(0, 1]$ can be cast in this form provided that n is large enough (see the appendix). A very efficient method for the computation of the steady state size distribution for kernels of the above type will be given next. The kernel substituted into the equation (7*b*) for the steady state, after some algebra, leads to

$$\bar{f}_s(\bar{x}) = \sum_{i=1}^{n} c_i x_m^{k_i+2} \int_{x_m}^{1} y^{-k_i-1} q(y)\,\mathrm{d}y \bar{x}^{k_i} + \sum_{i=1}^{n} c_i x_m^{k_i+2} \bar{x}^{k_i}.$$

This result can be simplified as

$$\bar{f}_s(\bar{x}) = \sum_{i=1}^{n} c_i \bar{x}^{k_i} [x_m^{k_i+2}(M_i + 1)] = \sum_{i=1}^{n} c_i \bar{x}^{k_i} F_i \tag{29}$$

where $M_i = \int_{x_m}^{1} y^{-k_i-1} \varphi(y)\,\mathrm{d}y$ and $F_i = x_m^{k_i+2}(M_i + 1)$.

It is obvious that the steady state size distribution has exactly the same form as the kernel. Consequently, any singularity of the kernel at $z = 0$ can be transferred analytically to the steady state size distribution. This fact is very important for singular kernels (even practical kernels can be singular at $z = 0$ [16]) for which the direct numerical solution for the steady state size distribution is difficult. Thus, the problem of computing the steady state distribution is reduced to determining the weights $F_i (i = 1, 2, \ldots, n)$ which are functions of x_m. To proceed in this way, the kernel is substituted in equation (7*a*). After some algebra (using the definition of M_i) the following result is obtained:

$$q(x) = \sum_{i=1}^{n} c_i x^{k_i} (M_i + 1). \tag{30}$$

Taking into account that $q(x) = -x^{k_i+1} \frac{\mathrm{d}M_i}{\mathrm{d}x}$, for $i = 1$ to n, a system of differential equations results

$$\frac{\mathrm{d}M_i}{\mathrm{d}x} = -\sum_{j=1}^{n} c_j x^{k_i+k_j-1} (M_j + 1) \qquad i = 1 \text{ to } n. \tag{31}$$

Finally, by employing the relation between M_i and F_i, and the timelike variable t, the system of differential equations for the F_i's is reduced to the following very simple form:

$$\frac{\mathrm{d}F_i}{\mathrm{d}t} = \sum_{j=1}^{n} c_j F_j - (k_i + 2) F_i \qquad i = 1 \text{ to } n. \tag{32}$$

For $t = 0$ the steady state distribution coincides with the kernel; thus the initial condition for the above system is $F_i(0) = 1$ for $i = 1$ to n. The weights of the limiting steady

state distribution $F_i(\infty)$ can be obtained from the solution of the system of linear algebraic equations that result by setting the derivatives in equations (32) equal to zero. The solution to this system is $F_i = U/(k_i + 2)$, where U is an arbitrary constant because the system is homogeneous. The constant U is specified to conserve the total mass of the system and the final result for the weights of the limiting steady state size distribution is given as

$$F_i = \frac{1}{k_i + 2}\left(\sum_{j=1}^{n} \frac{c_j}{(k_j + 2)^2}\right)^{-1}. \tag{33}$$

Exactly the same result is obtained by using equation (9) that directly provides the limiting steady state size distribution.

The system of linear differential equations with the given initial condition has the following solution [19]

$$F_i = \sum_{j=1}^{n} d_{ij} \mathrm{e}^{\lambda_j t} = \sum_{j=1}^{n} d_{ij} x_m^{-\lambda_j} \qquad \text{where } d_{ij} = \sum_{k=1}^{n} w_{ij} v_{jk} \tag{34}$$

$i = 1, \ldots n$ and $j = 1, \ldots n$.

In the above equations λ_i is the eigenvalues, w_{ij} are the elements of the eigencolumn matrix and v_{ij} are the elements of the eigenrow matrix, of a matrix $\boldsymbol{A}$ with elements $a_{ij} = c_i - (k_i + 2)\delta_{ij}$ where δ_{ij} is the Kronecker delta. An alternative way to solve the system is direct numerical integration. Advantages of the latter include that it is very easy to implement and the fact that a numerical integration reproduces all the steady states between the initial and final value of x_m.

For the significant case corresponding to $n = 2$ a very simple solution results

$$\begin{aligned} F_1 &= 1 + \frac{(x_m^{\alpha} - 1)}{\alpha}(k_1 + 2 - c_1 - c_2) \\ F_2 &= 1 + \frac{(x_m^{\alpha} - 1)}{\alpha}(k_2 + 2 - c_1 - c_2) \end{aligned} \tag{35}$$

where $\alpha = k_1 + k_2 + 4 - c_1 - c_2$.

Numerical results will be presented here for two general types of kernels, which are representative of binary breakage. These cases are a subset of the general kernels for multiple breakage given in [18]. The forms employed are simple substitutes for the two kernels (for binary breakage) widely used in practical cases, i.e. the normal distribution kernal and the U-shaped one [6].

(i) *Product kernel* $(m = 0, 1, 2, \ldots)$

$$\varphi(z) = \frac{2(2m + 1)!}{(m!)^2} z^m (1 - z)^m. \tag{36}$$

This kernel is uniform for $m = 0$. For other values of m it has a form similar to that of the well known normal distribution-type kernel [20–21]. The exponent m is related to the inverse of the standard deviation of the normal-shaped kernel. In the limit $m \to \infty$ this kernal tends to the equal size breakage form [22]. The interpretation of the above kernel in terms of the general form (28) gives $n = m + 1$, $k_i = m + i - 1$ and

$$c_i = \frac{2(-1)^{i-1}(2m + 1)!}{m!(i - 1)!(m - i + 1)!} \qquad \text{for } i = 1 \text{ to } m + 1.$$

(ii) *Sum kernel* $(m = 1, 2, 3, \ldots)$

$$\varphi(z) = (m + 1)[z^m + (1 - z)^m]. \tag{37}$$

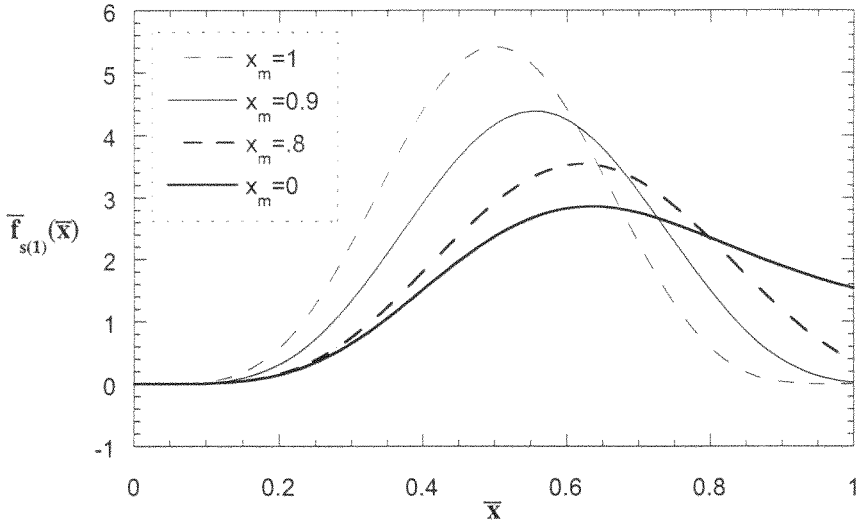

Figure 3. Steady state size distribution $\bar{f}_s(\bar{x})$ for the power-type kernel (equation (36)) with $m = 5$, for several x_m values.

This kernel is uniform for $m = 1$. For other values of m it has a U-shaped form. As m increases the kernel tends to represent an 'erosive' behaviour, i.e. preference for two daughter particles with very different sizes.

The interpretation of this kernel in terms of the general form (28) gives $n = m + 1$, $k_i = i - 1$, $c_i = \frac{(-1)^i(m+1)m!}{(i-1)!(m-i+1)!}$ for $i = 1$ to m and $c_{m+1} = (m + 1)(1 + (-1)^m)$.

Using the values of k_i and c_i for the above two kernels, the system of differential equations (32) is solved numerically using a Runge–Kutta integrator with prespecified accuracy and adjustable step [23]. The steady state size distributions for the product and sum kernels with $m = 5$ are shown in figures 3 and 4, respectively, for different x_m values. Interestingly, for the power kernel the limiting steady state is practically reached even for $x_m = 0.5$.

7. Approach to the limiting steady state

As outlined above, the steady state size distribution reaches an asymptotic form as x_m tends to zero which is called the limiting steady state. An interesting question is at which value of x_m the limiting steady state is attained. Here this issue is examined by employing the analysis of the previous sections for the steady state distribution. First, the perturbed cosine kernel is studied. A very sensitive measure for the deviation between 'transient' steady state and limiting steady state is the value at $\bar{x} = 1$, as can be seen in figures 1 and 2. The perturbation of the steady state is determined from equation (26) in the limit $\bar{x} = 1$

$$\bar{f}_{s(1)}(1) = \frac{8}{(n\pi)^2} - \frac{8\cos(n\pi x_m)}{(n\pi)^2} - \frac{8}{(n\pi)^2}\int_0^1 \frac{\cos(n\pi x) - 1}{x}\,\mathrm{d}x + 2x_m^2\cos(n\pi x_m). \qquad (38)$$

The above quantity is plotted in figure 5 versus the timelike variable $t = \log(1/x_m)$. These curves closely resemble the response of an overdamped dynamic system [24] to an external disturbance; i.e. with increasing n the amplitude of the oscillations decreases whereas the 'time', until the limiting steady state is reached, increases.

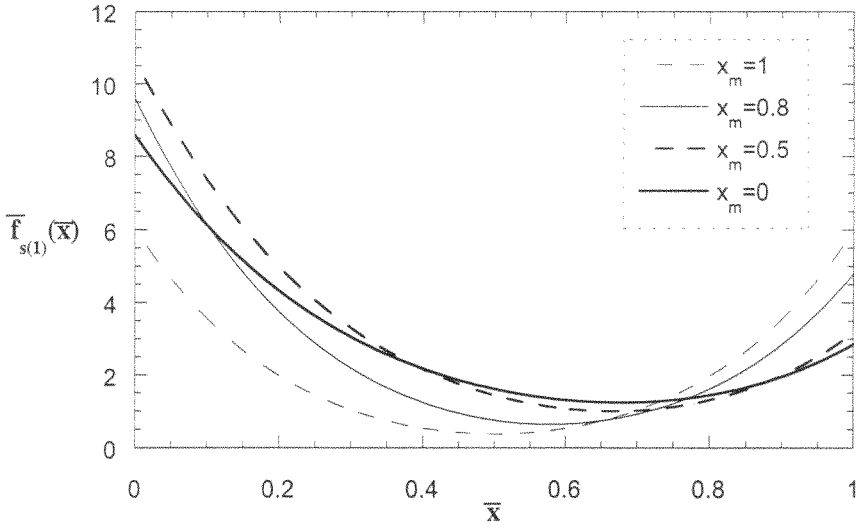

Figure 4. Steady state size distribution $\bar{f}(\bar{x})$ for the sum-type kernel (equation (37)) with $m = 5$, for several x_m values.

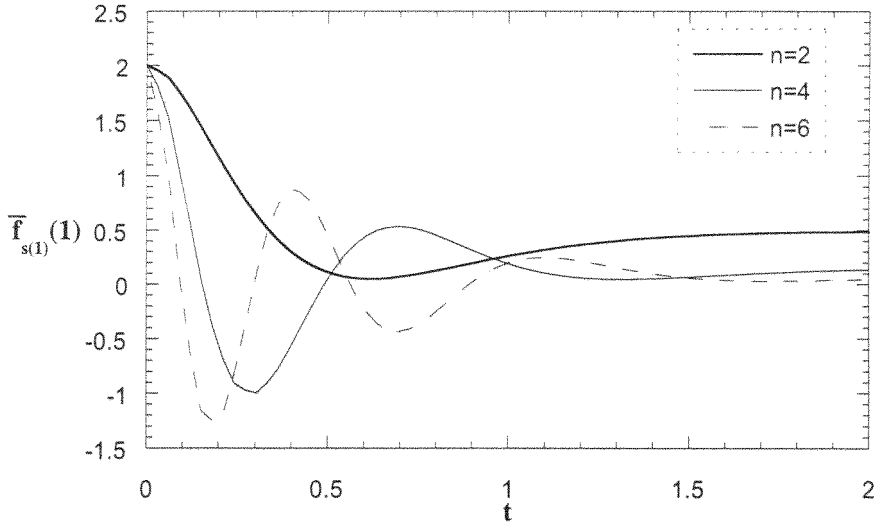

Figure 5. First-order perturbation of the steady state distribution $\bar{f}_{s(1)}$ (1) for an $n/2$ period cosine perturbation of the uniform kernel, versus artificial time for several n values.

In the following, the more general sum of powers kernel will be examined as regards the approach to the limiting steady state. As already discussed, the weights of the steady state distribution are given by a relation of the form

$$F_i = \sum_{k=1}^{n} d_{ik} x_m^{-\lambda_k}. \tag{39}$$

One of the eigenvalues λ_k of the matrix $\boldsymbol{A}$ must be zero (let us say $\lambda_1 = 0$) to account for the limiting steady state. On physical grounds it is expected that the remaining eigenvalues

Table 2. Exponent β of the approach to the limiting steady state for the generalized product kernel (equation (36)) for several m values.

m	β
1	7
2	7.5
3	6.5
4	5.5
5	4.75
6	4.15
7	3.66

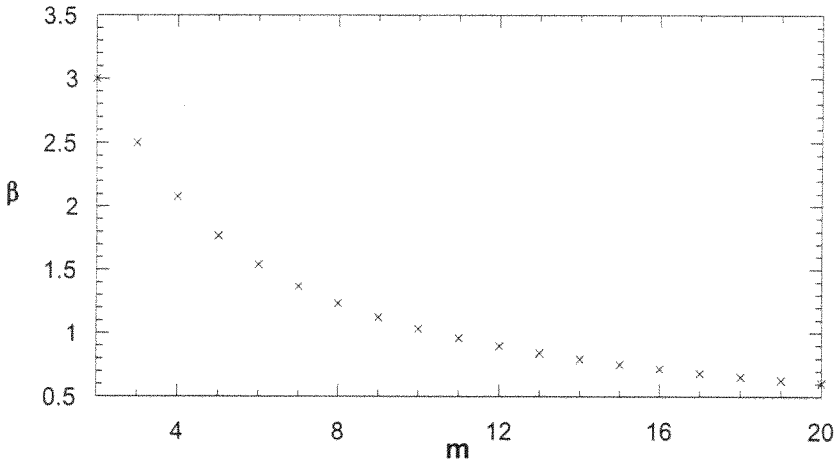

Figure 6. Exponent β of the approach to the limiting steady state for the generalized sum kernel (equation (37)) versus m.

would be real and negative. The approach to the steady state is determined by the smallest (in absolute value) eigenvalue. Summarizing, for the sum of powers kernel

$$f_s(\bar{x}) - f_{sl}(\bar{x}) \propto x_m^{\beta} \tag{40}$$

where $\beta = |\lambda_2|$ and λ_2 is the smallest, in absolute value, non-zero eigenvalue of the matrix $\boldsymbol{A}$. For example, let us explore the case of product kernel. The matrix $\boldsymbol{A}$ is constructed as outlined in the previous section. A QR decomposition algorithm [25] is used to compute the eigenvalues of matrix $\boldsymbol{A}$. The values of the exponent β for several m values are given in table 2. Due to the form of matrix $\boldsymbol{A}$ for this particular kernel (very large values of c_i) the algorithm with double precision becomes inaccurate for $m > 7$. This is why the direct numerical integration of the system of differential equations is indispensable in some cases. As can be seen in table 2, the exponent β is reduced with increasing m. The high value of β for $m = 5$ explains why the limiting steady state size distribution is reached even for $x_m = 0.5$ in the example shown in figure 5. In figure 6 the exponent β for the case of sum (erosion) kernel is plotted versus m. A nearly perfect fit to the points for $m > 10$ is $\beta = 6.132m^{-0.775}$. This relation can be used for the computation of the rate of approach exponent for very large m values (extremely 'erosion'-type kernels).

In both cases (product and sum kernels), with m increasing (kernel becoming steeper) the exponent β decreases. A generalization of this is the statement that as the kernel

becomes steeper (the maximum derivative increasing) the approach to the limiting steady state is 'delayed'. In the limit of a kernel with infinite steepness (discontinuous kernel) the exponent β tends to zero and consequently there is no limiting steady state size distribution, as is also pointed out elsewhere [7].

8. Comparisons with other methods

In this section the steady state size distribution obtained here will be compared with those resulting from analytical solutions of the *unlimited* breakage equation. If the solution of the unlimited breakage problem is $f(x,\tau)$, the steady state distribution for limited breakage is given as

$$\frac{\mathrm{d}f(x,\tau)}{\mathrm{d}\tau} = \int_{x_m}^{1} \frac{1}{y} b(y)\varphi(x/y) f(y,\tau)\,\mathrm{d}y \qquad \text{for } 0 < x < x_m \tag{41a}$$

$$\bar{f}_s(\bar{x}) = x_m^2 f(\bar{x}x_m, \infty). \tag{41b}$$

A well known analytical solution to the breakage problem exists for the simple power-law kernel (product kernel (36) for $m = 1$) [10]. In this case the steady state can be trivially obtained without resorting to the exact form of the analytical solution. Indeed, the power form of the steady state distribution is obvious from equations (41) whereas the coefficients of the power term can be determined simply by mass conservation considerations without employing $f(x,\tau)$.

A much more elaborate case [13] is for the kernel

$$\varphi(z) = \frac{\delta\lambda}{\delta - \lambda}(z^{\lambda-2} - z^{\delta-2}) \tag{42}$$

with $b(x) = x^{\lambda}$.

Obviously this kernel is of the sum of powers type (28) with $k_1 = \lambda - 2$, $k_2 = \delta - 2$, $c_1 = -c_2 = \delta\lambda/(\delta - \lambda)$. The closed-form transient solution with this kernel is [13]

$$f(x,\tau) = \mathrm{e}^{-\tau}\delta(x-1) + \lambda\delta\tau x^{\delta-2}\int_x^1 y^{\lambda-\delta-1}\mathrm{e}^{-\tau y^{\lambda}}\,\mathrm{d}y. \tag{43}$$

Substituting the above transient solution into equation (41*a*) and performing the integration from $\tau = 0$ to $\tau = \infty$, with the initial condition $f(x,0) = 0$, one obtains

$$f_s(x) = \frac{\lambda\delta}{\delta-\lambda}(x^{\lambda-2} - x^{\delta-2}) + \frac{(\lambda\delta)^2}{\delta-\lambda} \times\left[x^{\lambda-2}\int_{x_m}^{1}\int_z^1 z^{\delta-1}y^{-\lambda-\delta-1}\,\mathrm{d}y - x^{\delta-2}\int_{x_m}^{1}\int_z^1 z^{\lambda-1}y^{-\lambda-\delta-1}\,\mathrm{d}y\right]. \tag{44}$$

Carrying out the double integration and using relation (41*b*) leads to

$$\bar{f}_s(\bar{x}) = \frac{\lambda\delta}{\delta-\lambda}\left(\frac{\lambda x_m^{\lambda+\delta}}{\lambda+\delta} + \frac{\delta}{\lambda+\delta}\right)\bar{x}^{\lambda-2} - \frac{\lambda\delta}{\delta-\lambda}\left(\frac{\delta x_m^{\lambda+\delta}}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta}\right)\bar{x}^{\delta-2}. \tag{45}$$

This is exactly the result obtained independently with the direct method for the steady state distribution, i.e. equations (29) and (35) with the relevant values of k_1, k_2, c_1, c_2.

9. Conclusions

A theoretical study is presented of the steady state size distribution resulting from a limited breakage process with kernels homogeneous and continuous (except possibly at $x = 0$). It is shown that, due to the linearity of the problem, the steady state for an arbitrary initial distribution can be obtained by superposition of the steady states resulting from monodisperse initial conditions. The steady state distribution depends only on the kernel and the ratio of critical to initial size. As this ratio decreases, the steady state distribution evolves from its initial form (the same as the kernel) to the *limiting* one that has already been studied [7].

The asymptotic behaviour of the steady state distribution and of the auxiliary function used in its evaluation is given in this work for several limits. Knowledge of this behaviour can lead to a great simplification of the numerical solution for the steady state distribution. A perturbation analysis for nearly uniform kernels reveals features of the steady state distribution not accessible otherwise. It is shown that any continuous in $(0, 1]$ kernel can be cast in the 'sum of powers' form. In such a case the steady state distribution has the same form with the kernel but with a different weighting factor for each term of the sum. These factors are obtained from the solution of a homogeneous system of linear differential equations with constant coefficients. The analytical treatment presented herein allows one to study several interesting features of the problem; for example, the approach to the limiting steady state is found to be determined by the smallest eigenvalue of a certain matrix. Furthermore, with this treatment it is possible to obtain results for kernels so stiff that they cannot be handled numerically. Finally, it is shown that the direct computation of the steady state distribution provides exactly the same result as that obtained from the dynamic problem (as time goes to infinity) but in a more formal and easy way.

Appendix

Let $\varphi(x)$ be a kernel continuous in $(0, 1]$. In general $\varphi(x)$ has a singularity of order $\nu(\nu > -2)$ at $x = 0$. If $k \leqslant \nu$, one can write $\varphi(x) = x^k \xi(x)$ where $\xi(x)$ is now continuous in $[0, 1]$. The Weierstrass theorem [26] guarantees that there is a polynomial $P_n(x)$ of sufficiently high degree n that $|\xi(x) - P_n(x)| < \varepsilon$ for any value of x in $[0, 1]$, where ε is an arbitrarily small number. Thus, one can use the forms $\xi(x) = \sum_{i=0}^{n} a_i x^i$ and $\varphi(x) = \sum_{i=0}^{n} a_i x^{i+k}$. It is recognized that the general kernel $\varphi(x)$ is cast in the form of equation (28). The proper way for evaluating the coefficients a_i of this approximation is through the use of orthogonal polynomials. Especially for Legendre polynomials these coefficients are simple linear combinations of the moments of $\xi(x)$.

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